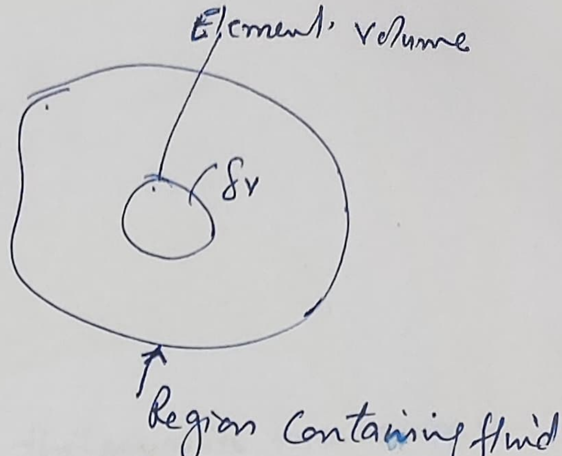
Fluid as a Continuation

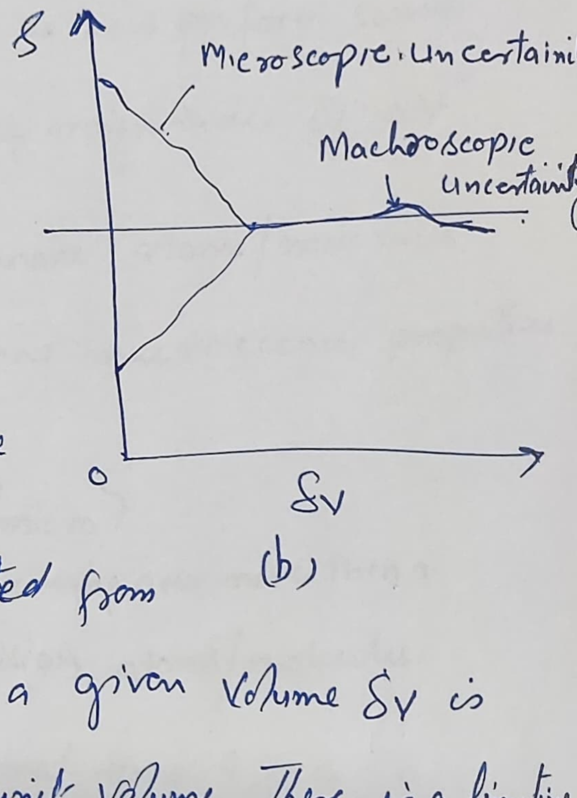


(a)

Figure shows the limit definition of continuum fluid density

a — An element volume in a fluid region of variable continuum density

b — Calculated density versus size of the element volume



(b)

where the density as calculated from molecular mass δm with in a given volume δv is plotted versus the size of the unit volume. There is a limiting volume δv^* below which molecular variations may be important and above which aggregate variations may be important. The density ρ of a fluid is best defined as

$$\rho = \lim_{\delta v \to \delta v^*} \frac{\delta m}{\delta v}$$

Therefore the smallest choice of $\Delta \forall$ that permits macroscopic property definition is $\Delta \forall^*$. Let us now perform some arithmetic to guage the order of magnitude of $\Delta \forall^*$.

Usually an ensemble of 10^6 or more atoms/molecules is considered suitable for defining macroscopic properties.

$$P_{macro} = \langle P_{micro} \rangle$$

↳ averaging over more than a million atoms/molecules.

The mean free path in air under normal temp. & pres. is around 10^{-8} m/10^{-5} mm. Therefore, in a cubic volume of $10^{-3}\,mm \times 10^{-3}\,mm \times 10^{-3}\,mm$, there would be $100 \times 100 \times 100 = 10^6$ atoms or molecules

$$\Rightarrow \Delta \forall^* \approx 10^{-9}\,mm^3 \text{ for air under Normal } T, P$$

Typically for liquids the mean free path is smaller than that for gases and $\Delta \forall^*$ therefore is even smaller than $10^{-9}\,mm^3$

T_2

I_2

A ⊙ — $\Delta V^* \cong 10^{-9}\ mm^3$

The atoms/molecules within ΔV^* contribute to the definition of macroscopic property at A.

Since the macroproperties can be defined at each point of the space occupied by the fluid, this leads to a continous distribution of macroscopic proputies in the fluid.

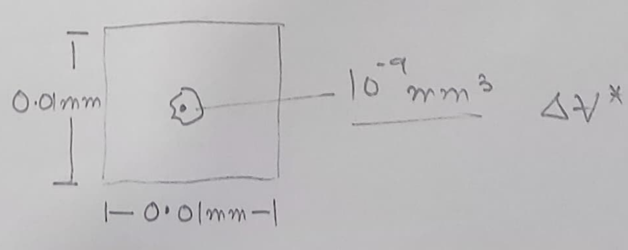
Treatment of fluid as a collective of continous distribution of macro properties is referred to as the concept of Continuum.

Validity of the continuum concept.

The validity of the continuum concept rests on the fact whether the size of the limiting ensemble ΔV^* is small enough to be treated as a point.

This in term depends on,

1) The mean free path, 'λ'
2) The physical size of the system under consideration.



Knudsen Number.

$$K_n = \frac{\lambda}{l} = \frac{\text{Mean free path}}{\text{System char. dimension}}$$

0.001 | 0.01 | 0.1 | 1 | 10 | 100 | 1000

Continuum concept is valid

Transitional region

rarefied fluid dynamics

Continuum concept with approximations

<u>Fluid particle</u>: The point like, limiting ensemble of atoms/molecules that is employed to define the macroscopic properties at a given point in a fluid.

\# The fluid can be viewed as an infinite set of fluid particles occupying different location, at a given instant of time.

\# If we can describe/analyse the motion of each and every fluid particle, then the motion of the entire fluid is described/analysed as a whole.

Force distribution in a fluid.

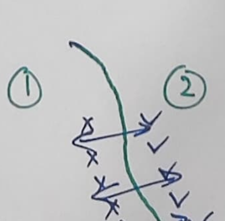
In mechanics we investigate the effect of forces on a medium.

In order to obtain working equations that reflect the behaviour of the media under the action of forces, a mathematical model (continuum) of the forces in the medium needs to be evolved.

In this context, we recognize two distinct origin of forces in a fluid.

1) Forces generated due to some external force field like, gravity, an electric field, a magnetic field etc. Such forces are termed as body forces. Their effect is distributed over entire mass or volume.

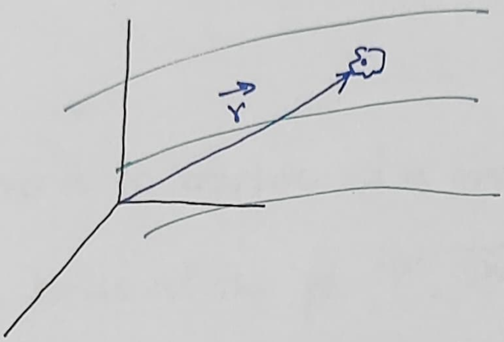
2) Forces that are generated between the fluid particles as an outcome of intermolecular exchanges of momentum. Such forces are visualised in the form of interactions between fluid particles on either side of an imaginary surface. Such forces are termed as surface forces.



Body forces

Let the body force intensity/mass be specified as $\vec{B}(r,t)$.

Since, the body forces are due to some externally imposed force field, generally the intensity is known. independent of the state of the fluid.



The figure shows a fluid particle at $(\vec{r})$ at some time 't'

The body force on the particle can be expressed as

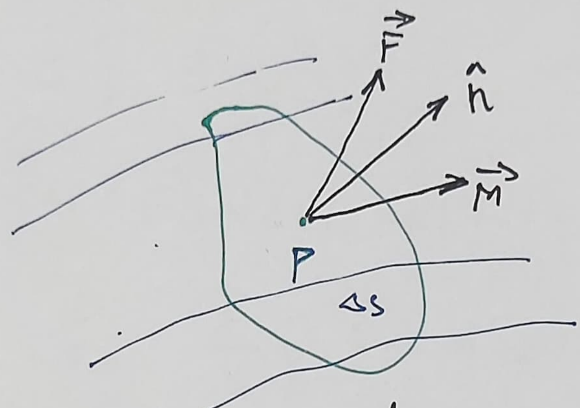
$B(\bar{r},t)\,\rho\,d\forall$ where ρ is the density of the fluid particle and $d\forall$, the infinitesimal volume of the fluid particle. (Both ρ and $d\forall$ are defined in the continuum limiting sense)

If a finite region is considered, then the total body force on the fluid region (system) can be expressed as:

$$\bar{F}_B = \int_{VOL} \bar{B}(\bar{r},t)\,\rho\,d\forall$$

Surface force

Consider a pt 'P' lying on an arbitrary planar surface. which is identified by its normal ($\hat{n}$)



The objective is to arrive at a mathematical description of surface forces at the pt 'P'. The surface forces must be acting at each and every point of the surface in different directions. However, we know from previous Applied Mechanics course that a system of distributed forces can always be reduced to a single ferce $\vec{F}$ and a moment $\vec{M}$ at the given point 'P'.

Since the objective is to arrive at a state of surface forces at the pt 'P', we must investigate the effect of reducing the surface area 'Δs' on the force $\vec{F}$ and the moment $\vec{M}$.

As $\Delta s \to 0$, we recognise that the surface force that are in general varying in magnitude as well as direction over the surface would tend to become more and more uniform as the area reduces.

Thus, it may be argued that as

$\Delta S \to 0$, $|\vec{F}| \propto \Delta S$, and $\vec{M} \to 0$

further as $|\vec{F}| \propto \Delta S$, the limit

$$\underset{\Delta S \to 0}{Lt} \frac{\vec{F}}{\Delta S} := \vec{\sigma_n} \text{ is well defined.}$$

Thus the effect of surface forces at a pt 'P' lying on a given planar surface (normal $\hat{n}$) can be represented by the area intensity $\vec{\sigma_n}$, which is the stress vector at a pt 'P'.

The above mathematical representation of the effect of surface forces at 'P' still involves arbitrary choice of a plane surface through P. It is expected that a different choice of a planar surface through P would lead to a different stress vector $\vec{\sigma_n}$ at pt. 'P'

In order to overcome this problem, stress vectors at P on three coordinate planes through P are specified.

This specification is sufficient to determine the stress vector at P on any plane through P.

At any pt. we specify

$$\vec{\sigma_x} = \sigma_{xx}\hat{i} + \sigma_{xy}\hat{j} + \sigma_{xz}\hat{k}$$
$$\vec{\sigma_y} = \sigma_{yx}\hat{i} + \sigma_{yy}\hat{j} + \sigma_{yz}\hat{k}$$
$$\vec{\sigma_z} = \sigma_{zx}\hat{i} + \sigma_{zy}\hat{j} + \sigma_{zz}\hat{k}$$

The stress vector $\vec{\sigma_n}$ at the same pt. on any plane whose normal is $\hat{n}$ having direction coeff.'s (l, m, n) can be found as follows

$$\begin{pmatrix} \sigma_{nx} \\ \sigma_{ny} \\ \sigma_{nz} \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} \quad - (1)$$

Fluid Statics.

A fluid can not withstand a slightest shear stress. Even an infinitesimal shear introduces a deformation that is continous in time. This is in contrast to the case of solids where the deformation is finite and takes place in a very short time interval.

From the above it can be concluded that in a stationary fluid, at any pt, there can be no shear stress on any plane passing through the pt. Thus at any pt, all planes are principal planes.

Thus the stress state at a point is given as

$$\begin{pmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix}$$

Further it can be shown that the principal stress (normal stress) on each and every plane passing through a given pt. is same.

This is done as follows.

Consider an arbitrary plane having dc's (l, m, n) through a pt.

$$\begin{pmatrix} \sigma_{nx} \\ \sigma_{ny} \\ \sigma_{nz} \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & & \\ & \sigma_{yy} & \\ & & \sigma_{zz} \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} \quad —$$

$$\Rightarrow \sigma_{nx} = \sigma_{xx} l, \quad \sigma_{ny} = \sigma_{yy} m, \quad \sigma_{nz} = \sigma_{zz} n \quad — (1)$$

Denoting the normal stress on the plane as σ_{nn}, we have.

As the plane is Principal plane

$$\left\{ \begin{aligned} \sigma_{nx} &= l\,\sigma_{nn} \\ \sigma_{ny} &= m\,\sigma_{nn} \\ \sigma_{nz} &= n\,\sigma_{nn} \end{aligned} \right. \quad — (2)$$

Comparing (1) & (2)

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{nn} = \sigma \text{ (say)} \quad - (3)$$

Stress state is isotropic.

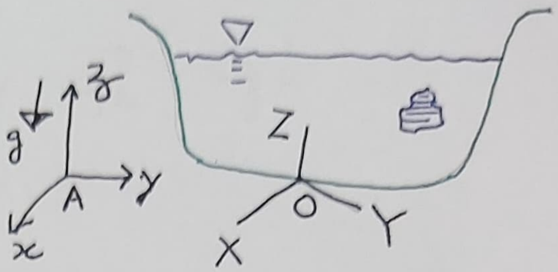
It is found that the normal stress σ at a point is always compressive in character in a stationary fluid.

$$\Rightarrow \sigma = -\underset{\to \text{ fluid pressure}}{p}$$

Accelerating Containers: Rigid body motion.

Suppose the vessel is imparted a rectilinear uniform accn. $\vec{a}$.

Initially, the fluid in the vessel undergoes some oscillations (sloshing) but after a short period the fluid attains a state of rest w.r.t the vessel.



$$\Rightarrow \quad \vec{V}_{XYZ} = \vec{a}_{XYZ} = 0.$$

Applying Newton's 2nd Law to the fluid mass (shown hatched) using the XYZ frame.

$$\oiint -p\hat{n}\,ds + \iiint \rho \vec{f}_b \,d\forall = \iiint \rho \left(\vec{a}_{XYZ} + \vec{a}_{O/A}\right) d\forall$$

$$\vec{a}_{O/A} = \vec{a}$$

$$\oiint -p\hat{n}\,ds + \iiint \rho \vec{f}_b \,d\forall = \iiint \rho \vec{a}\,d\forall$$

$$\Rightarrow \int_{vol} \left(-\nabla p + \rho \vec{f}_b\right) d\forall = \int_{vol} \rho \vec{a}\,d\forall$$

$$\text{or} \quad \int_{vol} \left(-\vec{\nabla} p + \rho \vec{f}_b - \rho \vec{a}\right) d\forall = 0$$

Since limits are arbitrary $\Rightarrow -\vec{\nabla} p + \rho \vec{f}_b = \rho \vec{a}$

$$\vec{\nabla} p = \rho \underbrace{\left(\vec{f}_b - \vec{a}\right)}_{\text{effective body force}} \quad \text{— (1)}$$

An important consequence of eq ① is that isobaric surfaces are not $\perp$ to the body force, but to the effective body force.

Now body force due to gravity : $\vec{f_b} = -g\hat{k}$

$$\vec{\nabla} p = \rho(-g\hat{k} - \vec{a})$$

Choosing the XYZ, such that $\vec{a}$ lies in one of the coordinate planes.

$$\vec{a} = a_x\hat{i} + a_z\hat{k}$$

$$\Rightarrow \vec{\nabla} p = \rho(-g\hat{k} - a_x\hat{i} - a_z\hat{k})$$

$$= -\rho(a_x\hat{i} + (g+a_z)\hat{k})$$

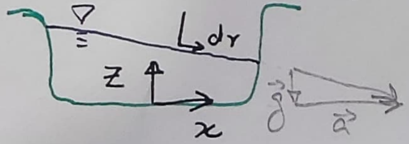
$$\therefore \frac{\partial p}{\partial x} = -\rho a_x, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = -\rho(a_z+g) \quad \Rightarrow p \equiv p(x,z)$$

$$\frac{\partial p}{\partial x} = -\rho a_x, \Rightarrow p(x,z) = -\rho a_x x + f(z)$$

$$\frac{\partial p}{\partial z} = f'(z) = -\rho(a_z + g)$$

Integrating, $f(z) = -\rho(a_z+g)z + A$

$$\Rightarrow p(x,z) = -\rho a_x x - \rho(a_z+g)z + A$$



R3

Slope of free surface (Any isobaric surface)

$$\frac{dz}{dx} = \frac{-\partial p/\partial x}{\frac{\partial p}{\partial z}} = \frac{-\rho a_x}{\rho(a_z+g)}$$

$\vec{\nabla} p \cdot d\vec{r} = 0$

$\vec{\nabla} p$: normal to the surface

$d\vec{r}$: tangent vector to the isobaric surface

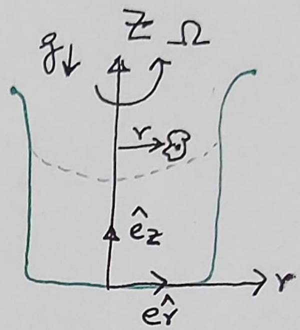
free surface

$$dp(x,z) = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial z}dz$$

$dp = 0$

$$\frac{dz}{dx} = -\frac{\partial p/\partial x}{\partial p/\partial z}$$

Rotating Vessel



$\therefore -\vec{\nabla} p + \rho \vec{f}_B = \rho \vec{a}$ — from equ. 1

$\vec{a} = -\omega^2 r \hat{e}_r$

$\vec{f}_B = -g\hat{e}_z$

$\vec{\nabla} p = \rho(-g\hat{e}_z + \omega^2 r \hat{e}_r)$

$\vec{\nabla} p = \hat{e}_r \frac{\partial p}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial p}{\partial \theta} + \hat{e}_z \frac{\partial p}{\partial z}$

$\frac{\partial p}{\partial r} = \rho\omega^2 r$

$\frac{\partial p}{\partial z} = -\rho g$

$\frac{\partial p}{\partial \theta} = 0$

$$\Rightarrow p(z,r) = \frac{\rho\omega^2 r^2}{2} - \rho g z + A$$

For finding free surface (Isobaric surface)

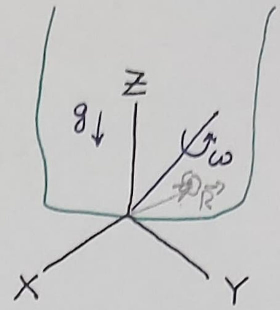
$$\vec{\nabla} p \cdot d\vec{r} = 0 \;;\; \frac{dz}{dr} = \frac{-\frac{\partial p}{\partial r}}{\frac{\partial p}{\partial z}} = \frac{\omega^2 r}{g} \quad \text{or } z = \frac{\omega^2 r^2}{2g} + B$$

$\Downarrow$

eq. of parabola

Arbitrary Rotation Axis.

For a fluid contained in a sealed container, rotation axis can be arbitrarily oriented.



$$-\bar{\nabla}p + \rho\bar{g} = \rho\overbrace{[\bar{\omega}\times\bar{\omega}\times\bar{r}]}^{\bar{a}} \quad -(1)$$

$$\bar{g} = -g\hat{k}, \quad \bar{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k} \quad -(2)$$

$$\bar{\omega}\times\bar{r} = (\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k})\times(X\hat{i} + Y\hat{j} + Z\hat{k})$$

$$= Y\omega_x\hat{k} - \omega_x Z\hat{j} - X\omega_y\hat{k} + Z\omega_y\hat{i} + X\omega_z\hat{j} - Y\omega_z\hat{i}$$

$$= (Z\omega_y - Y\omega_z)\hat{i} + (X\omega_z - Z\omega_x)\hat{j} + (Y\omega_x - X\omega_y)\hat{k} \quad -(3)$$

$$\bar{\omega}\times(\bar{\omega}\times\bar{r}) = (\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k})\times(\quad\cdots\quad)$$

$$= (X\omega_x\omega_z - Z\omega_x^2)\hat{k} - (Y\omega_x^2 - X\omega_x\omega_y)\hat{j}$$

$$-(Z\omega_y^2 - Y\omega_y\omega_z)\hat{k} + (Y\omega_x\omega_y - X\omega_y^2)\hat{i}$$

$$+(Z\omega_y\omega_z - Y\omega_z^2)\hat{j} - (X\omega_z^2 - Z\omega_x\omega_z)\hat{i}$$

$$= [-X(\omega_y^2 + \omega_z^2) + Y\omega_x\omega_y + Z\omega_x\omega_z]\hat{i} +$$

$$[X\omega_x\omega_y - Y(\omega_x^2 + \omega_z^2) + Z\omega_y\omega_z]\hat{j} +$$

$$[X\omega_x\omega_z + Y\omega_y\omega_z - Z(\omega_x^2 + \omega_y^2)]\hat{k} \quad -(4)$$

From (1) and (4) we get.

$$\bar{\nabla}p = \rho(\bar{g} - \bar{a})$$

$$= \rho[X(\omega_y^2 + \omega_z^2) - Y\omega_x\omega_y - Z\omega_x\omega_z]\hat{i}$$

$$+ \rho[Y(\omega_x^2 + \omega_z^2) - X\omega_x\omega_y - Z\omega_y\omega_z]\hat{j}$$

$$+ \rho[-g + Z(\omega_x^2 + \omega_y^2) - X\omega_x\omega_z - Y\omega_y\omega_z]\hat{k} \quad -(5)$$

If rotation axis concides with $\bar{g}$

$$\bar{\omega} = \pm\omega_z\hat{k} \quad -(6)$$

$$\bar{\nabla} p = \rho[X\omega_z^2]\hat{i} + \rho[Y\omega_z^2]\hat{j} + \rho[-g]\hat{k} \quad -(7)$$

$$\frac{\partial p}{\partial x} = \rho X\omega_z^2 \,, \quad \frac{\partial p}{\partial y} = \rho Y\omega_z^2 \,, \quad \frac{\partial p}{\partial z} = -\rho g \quad -(8)$$

$$p = \frac{\rho X^2\omega_z^2}{2} + f(Y,Z) \quad -(9)$$

$$\frac{\partial p}{\partial Y} = \frac{\partial f}{\partial Y} = \rho Y\omega_z^2 \quad - 10$$

$$f(Y,Z) = \frac{\rho Y^2\omega_z^2}{2} + g(z) \quad - 11$$

$$\frac{\partial f}{\partial z} = \frac{\partial p}{\partial z} = g'(z) = -\rho g \,; \quad - 12$$

$$g(z) = -\rho g Z + C \quad -(13)$$

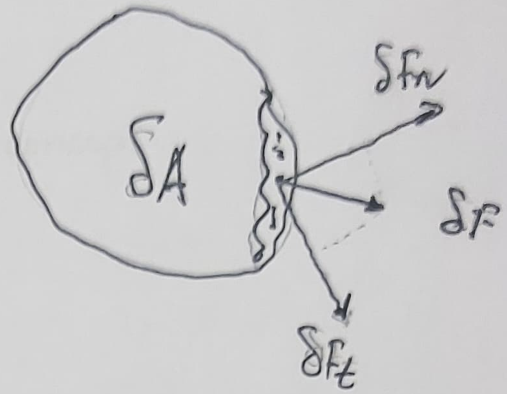
$$p(X,Y,Z) = \frac{1}{2}\rho\omega_z^2(X^2+Y^2) - \rho g z + C$$

Concepts

1. Stress.

Normal stress

$$\sigma = \underset{\delta A \to 0}{Lt} \left(\frac{\delta F_n}{\delta A} \right)$$



Shear Stress

$$\tau = \underset{\delta A \to 0}{Lt} \left(\frac{\delta F_t}{\delta A} \right)$$

2. Solids vs Fluids.

1. Molecular Structure
2. Behaviour in the presence of external applied force. (Shear force).

3. Concept of Continuum.

1. Continuous distribution of fluid properties. e.g. mass; no empty spaces.
2. Properties obtained at macro level are ensemble average of ~~micro level~~ prop at micro level.

Reading :-

Ch 1 :- The fluid as a continuum.

3.1. Validity of continuum concept.

$$Kn = \frac{\lambda}{L}$$

Knudsen No :-

λ: mean free path, avg. distance a molecule travels between successive collision.

L: Size of the system under consideration.

$Kn < 0.01$ Continuum concepts hold good.

$Kn > 10$ Need for Statistical mechanics.

eg. Rarefied gases.

Note! Using statistical/Quantum mechanics macro properties can be derived.

↳ Higher level course (Grad level).

4. Thermodynamic properties of a fluid. (Reading)

1. Pressure (P)   2. Density (ρ)   3. Temperature (T)

4. Velocity (V)

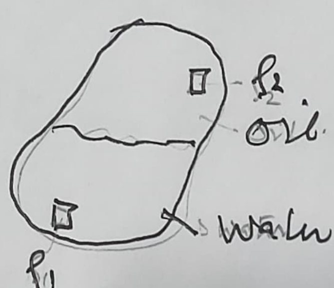
Primary therm. Variables.

Then there are two transport properties.

5. Coeff of viscosity, μ 9. Thermal conductivity (k)

NOTE: All of the above fluid properties are considered at a point. or at a limiting volume. (in the cont. limit)

$$\rho = \frac{\delta m}{\delta V}$$



5. Viscosity:-

1. Effect is understood when fluid is in motion.
2. When fluid elements move with diff. velo they feel resistance

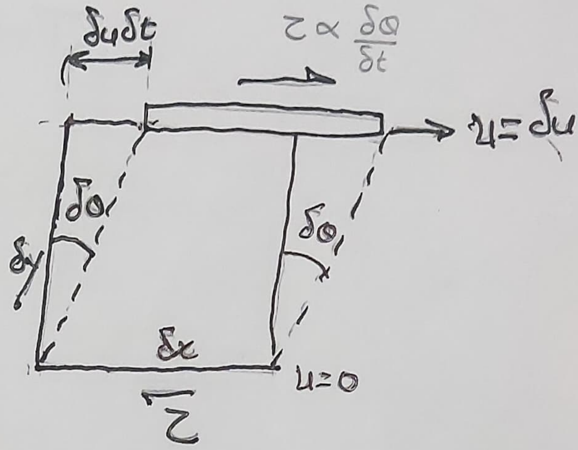
~~3. This resistance or the shear stress.~~

6. Newton's Law of Viscosity.

Sir Issac Newton (1687)

1. Shear Stress $\propto$ Shear strain rate.

$$\tau \propto \frac{\delta\theta}{\delta t}$$

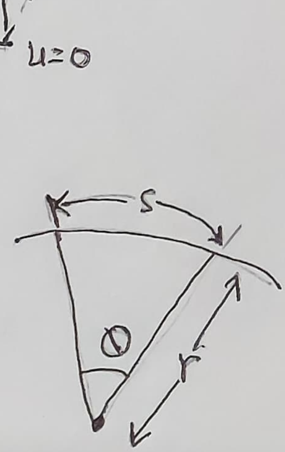


$$\tan\delta\theta = \frac{\delta u\,\delta t}{\delta y}$$

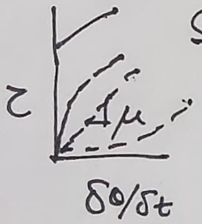
Use arc Length relation

$$S = r\theta$$

Alternatively, this can be used.



For $\delta\theta$, δu, δt and δy to be infinitesimally Small.



$$d\theta = \frac{du\,dt}{dy} \quad \text{or} \quad \frac{d\theta}{dt} = \frac{du}{dy}.$$

So,

$$\tau \propto \frac{du}{dy}$$

Dimensions of μ ??

$$= \mu \frac{du}{dy}$$

$y = mx$

; μ is the constant of proportionality.

Note: Fluids which follow above relation are called Newtonian fluids.

7. Comments about μ.

$$\mu = \mu(P,T)$$

⇒ μ is weakly affected by P

⇒ Strongly affected by T

For T ↑ ; μ_{gas} ↑ ?? Why.

μ_{Liq} ↓

7.1 Causes of viscosity.

a) Intermolecular forces of cohesion.

b). Molecular momentum exchange

→ In Liq. molecular motion is less significant

As, T ↑, molecular cohesion ↓.

→ For gas, molecular motion is more significant.

As T ↑, molecular motion ↑

Linear Transport Laws.

1. $\vec{q_m} = -K_m \nabla C$; $\vec{q_m}$: mass flux vector. $\frac{kg}{m^2 s}$; ∇C is concentration gradient.

2. $q = -k \nabla T$; $\vec{q}$ is heat flux vector. $\frac{J}{m^2 s}$

3. $\tau = \mu \frac{du}{dy}$;

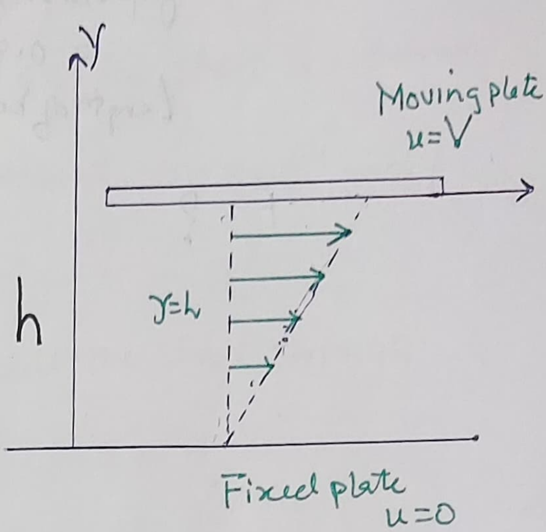
Only the 1st derivative of ~~the lin~~ some generalised concentration appears on RHS. This is because transport is carried out by molecular processes, in which length scales (say, mean free path) are too small to feel the curvature of the C-profile. Also the non linear terms involving higher powers of ∇C do not appear. This is expected for small values of ∇C only.

Although m & q are scalars but their flux are vector. But for τ eq. ~~the~~ u is vector & τ is tensor.

Coming back:

$$\tau = \mu \frac{du}{dy}$$

or $\frac{du}{dy} = \frac{\tau}{\mu} = $ constt.



Integrating above eq.

$$\int du = \int b\,dy$$

$$u = a + by$$

Using no slip B.C.

$u = 0$; $y = 0$

$= V$; $y = h$

$\therefore$ $u = \frac{Vy}{h}$ or $\frac{u}{y} = \frac{V}{h}$

Dynamic Viscosity: $\mu =$ $Kg/m.s$

Kinematic viscosity: $\nu = \frac{\mu}{\rho}$ m^2/s

Dynamic viscosity is the force per unit area required to ~~move~~ produce a relative motion of unit velocity between two fluid planes separated by unit distance

Although $\tau \propto \mu$, the tendency of a fluid to diffuse velocity gradients is determined by ν.

Given: Shaft: $d = 25mm$

gap b/w shaft & bearing

$= 0.3 mm.$

Length of bearing $= 0.5 m.$

Lube: $\nu = 8 \times 10^{-4} m^2/s$

$\rho = 910 \, kg/m^3$

$F = ?$ $\quad V = 3 m/s$

Problem! - 1. (7)

A 25 mm shaft is being pulled through a bearing filled with lubricant. The gap b/w the bearing and the shaft is 0.3 mm and the bearing is 0.5 m long. Lubricant, $\nu = 8 \times 10^{-4}\ m^2/s$ and $\rho = 910\ kg/m^3$.

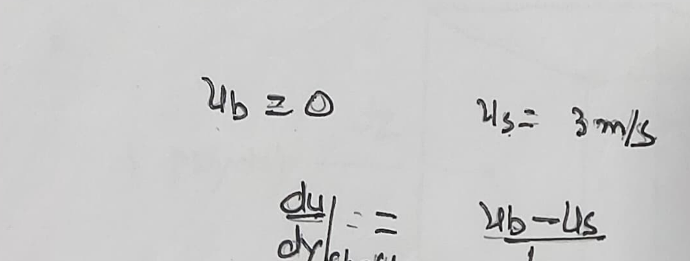
Find the force necessary to pull the shaft through the bearing at 3 m/s.

Assumptions:-

a) Lubricant is newtonian fluid.

b) Velo profile in the gap is linear.

c) No-slip B.C at the solid boundaries.



$u_b = 0$ $\qquad u_s = 3\ m/s$

$$\left.\frac{du}{dy}\right|_{shaft} = \frac{u_b - u_s}{h}$$

Using $\mu = \rho\nu$, $\tau = F/A$. A is area of shaft in bearing.

$$\tau = \mu\frac{du}{dy} = \frac{F}{A} = \rho\nu\left(\frac{u_b - u_s}{h}\right)$$ Find units of F??

$$\therefore F = 2\pi \underset{25\ mm}{R} \overset{0.5}{L} \rho\nu \frac{u_b - u_s}{\underset{0.3\ mm}{h}} \simeq -286N.$$ (exerted by lub. towards left)

Pulling force = +286 N.

Torque to be applied to get ω.

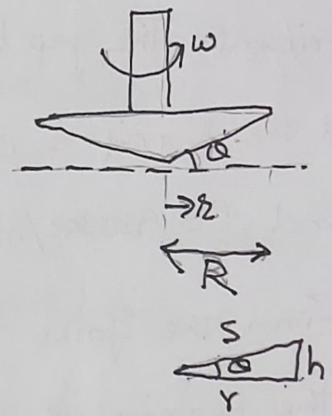
$$dT = dF \times r$$

$$= \tau \, dA \times r$$

$$= \left[\mu \frac{du}{dy}\right]\left[2\pi s \, dr\right] r.$$

$$= \left[\frac{\mu \omega r}{r \tan\theta}\right]\left[2\pi \frac{r}{\cos\theta} dr\right] r$$

$$T = \int_0^R \left[\frac{\mu \omega r}{r \tan\theta}\right]\left[\frac{2\pi r \, dr}{\cos\theta}\right] r.$$



Fluid Statics:-

1. Two types of forces act on a fluid element.

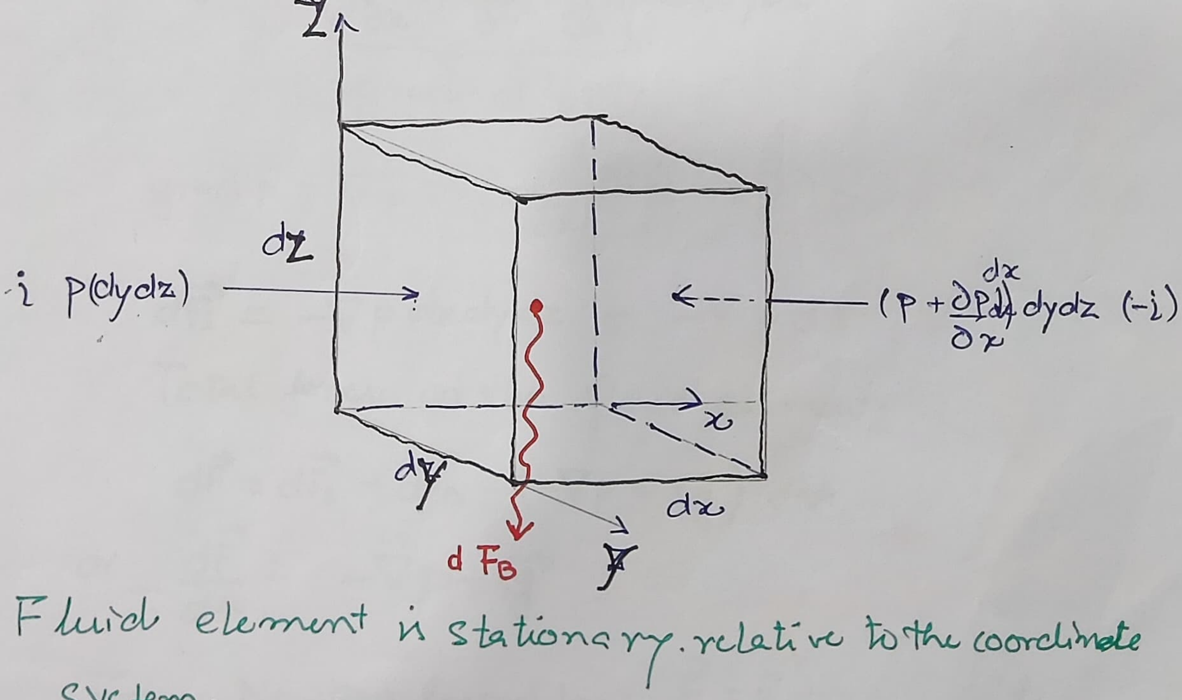
a) Body force: Distributed over entire mass or volume.
eg. Grav. force, E.Magnetic force.

b) Surface force:- Forces exerted on the fluid element at the surface.

They have two compoments.

- Normal force: Normal to surface
- Shear force: Tangential to surface.

2. Let us consider a differential fluid element.



Fluid element is stationary relative to the coordinate system.

Body force for the fluid element.

arrow denotes vector. It's a convention $d\vec{F_B} = \vec{g}\,dm = \vec{g}\,\rho\,d\forall$

$= \rho\vec{g}\,dx\,dy\,dz$

$d\forall$ is the volume of the fluid element

$d\forall = dx\,dy\,dz$.

\# Fluid is static $\Rightarrow$ no shear stress.

\# Only surface force is the pressure force.

\# Pressure is a scalar field $p = p(x, y, z)$.

From previous fig. we could find net force in x-dir

$$d\vec{F_x} = \hat{i}\,p\,dy\,dz - \hat{i}\left(p + \frac{\partial p}{\partial x}dx\right)dy\,dz = -\frac{\partial p}{\partial x}dx\,dy\,dz\,\hat{i}$$

Similarly $d\vec{F_y} = -\frac{\partial p}{\partial y}dx\,dy\,dz\,\hat{j}$; $d\vec{F_z} = -\frac{\partial p}{\partial z}dx\,dy\,dz\,\hat{k}$

$$d\vec{F_s} = -\underbrace{\left(\frac{\partial p}{\partial x}i + \frac{\partial p}{\partial y}j + \frac{\partial p}{\partial z}k\right)}_{\text{gradient of pr.}}dx\,dy\,dz$$

$$\text{grad } p \equiv \nabla p = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)p$$

$$d\vec{F_s} = -\nabla p\,dx\,dy\,dz = -\nabla p\,d\forall$$

Total force on the fluid element.

$$d\vec{F} = d\vec{F_s} + d\vec{F_B} = (-\nabla p + \rho\vec{g})\,d\forall$$

or $$\frac{d\vec{F}}{d\forall} = -\vec{\nabla}p + \rho\vec{g}$$

From Newton's Second law, $\vec{F} = \vec{a}\,dm = \vec{a}\,\rho\,d\forall$.

For a static fluid, or fluid at $V = $ const, $\vec{a} = 0$

$$\therefore \frac{d\vec{F}}{d\forall} = \rho\vec{a} = 0$$

$$\therefore -\vec{\nabla}p + \rho\vec{g} = 0$$

Net pr. force/volume at a point. + Body force/volume at a point = 0

Above eq. is a vector eq. So component eq.

$$-\frac{\partial p}{\partial x} + \rho g_x = 0 \qquad x\text{-dir}$$

$$-\frac{\partial p}{\partial y} + \rho g_y = 0 \qquad y\text{-dir}$$

$$-\frac{\partial p}{\partial z} + \rho g_z = 0 \qquad z\text{-dir}$$

In the chosen coordinate sys. gravity force is aligned ~~along~~ with z axis.

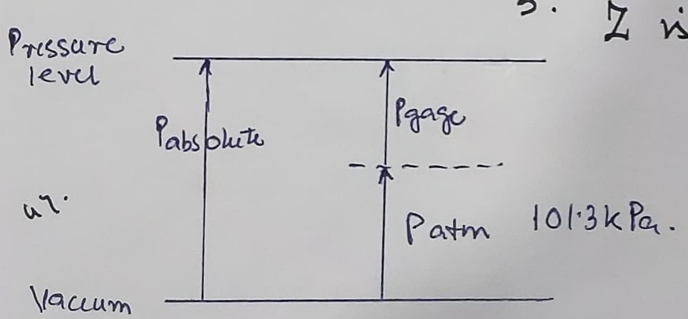
$$\Rightarrow \quad g_x = g_y = 0, \quad g_z = -g \qquad (Z \text{ is +ve upwards})$$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = -\rho g$$

or $\dfrac{dp}{dz} = -\rho g$ (Since p depends on z only so no need of partial derivative)

Restrictions:-

1. Static fluid
2. Gravity is the only F_B
3. Z is vertical & upwards.



For incompressible liq. & no g changes.

$\rho g \equiv$ constant.

$$\int \frac{dP}{dz} = \int -\rho g$$

or $$P_2 - P_1 = \rho g (z_1 - z_2).$$

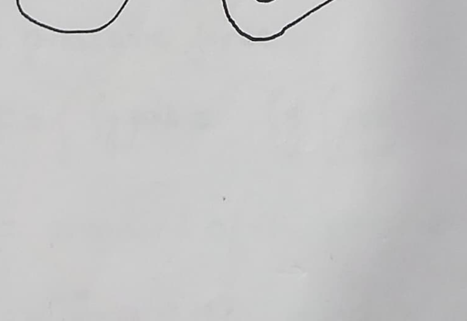
Easy to remember.

$$P_{down} = P_{up} + \rho g (\Delta z)$$

Applications. 1868

1. Mercury Barometer. (Abs. Pr)
2. Manometer (Diff. Pr)

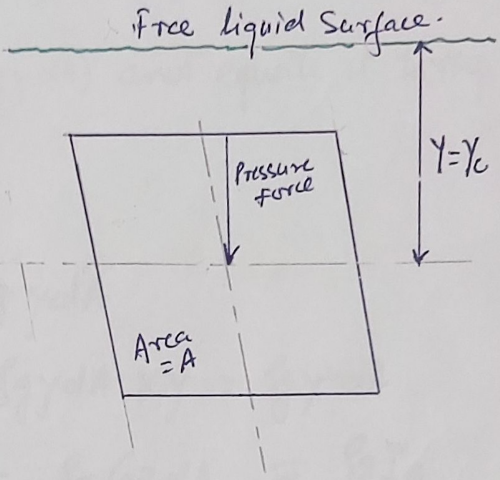
Inclined tube manometer [To increase sensitivity)



Hydrostatic force on a Plane Surface.

1. On a Horizontal Surface. (Submerged)

$F = pA = \rho g y A$



For a horizontal surface

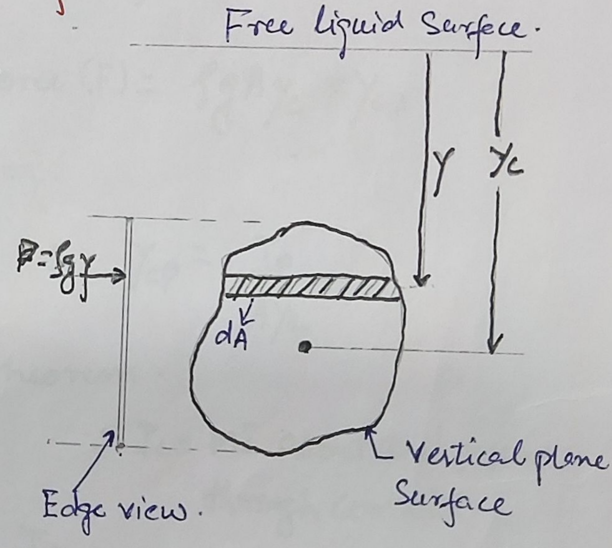
$y = y_c$.

$\therefore F = \rho g y_c A$

2. On a submerged vertical surface.

Pressure varies from top to bottom.

Consider a strip of area dA at a depth of y.



Pressure on strip $= \rho g y$

Pressure force on strip

$dF = p\,dA = \rho g y\,dA$

Total pressure force

$\int dF = \int \rho g y\,dA = \rho g \underline{\int y\,dA}$.

$\int y\,dA$ = moment of the surface about the free liquid surface.

= Area of the Surface × distance of c.g from free liq. surf.

= $A \times y_c$

$\therefore F = \rho g A y_c$

(x_{cp}, y_{cp})
Centre of Pressure: The point of application of the total pr. force on the surface.

For finding the location of (x_{cp}, y_{cp}) we find the sum of the moments of the elemental force ($\rho g y \, dA$) and equate it to the moment of the resultant force.

Pressure force on strip $= dF = \rho g y \, dA$

Moment about free liq. surf. $= \rho g y \, dA \times y = \rho g y^2 dA$

Sum of moments $= \int \rho g y^2 dA = \rho g \int y^2 dA = \rho g I_0$

But $\int y^2 dA =$ M. Inertia of the surface (entire) about axis. $= I_0$

Moment due to resultant force $(F) = \rho g A y_c \times y_{cp}$

Equating the above two eq.

$$\rho g A y_c \times y_{cp} = \rho g I_0 \qquad \therefore \; y_{cp} = \frac{I_0}{A y_c}$$

From parallel Axis theorem.

$$I_0 = I_c + A y_c^2$$

$\because I_c =$ MI about axis passing through centroid.

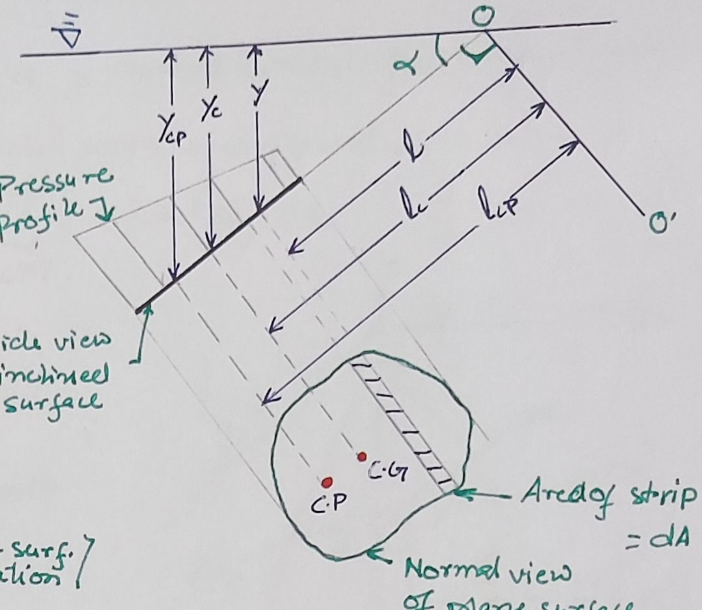
$$\therefore \; y_{cp} = \frac{I_c + A y_c^2}{A y_c} = \frac{I_c}{A y_c} + y_c$$

$$\text{or } y_{cp} - y_c = \frac{I_c}{A y_c}$$

Comments:

1. Centre of pr. lies below centroid as I_c/Ay_c is +ve.
2. y_{cp} is not a function of $\rho_{liq} \times g$. i.e y_{cp} is same for all liquids.

(3) On an inclined plane.



Force on elemental area (dA)

$dF = \rho g y\, dA$

$\quad = \rho g l \sin\alpha\, dA$

Force on entire surface

$$F = \int dF = \rho g \sin\alpha \int l\, dA$$

We know from before [Vertical surf. derivation]

$$\int l\, dA = A l_c.$$

$$\therefore\ F = \rho g l_c \sin\alpha\, A = \rho g A y_c = \text{Pr. at centroid} \times \text{Area (total)}$$

Note:- Force (F) is independent of the inclination angle.

For finding C.P location we take moment of both the elemental & the resultant pr. force about axis O–O'. and would equate them by integrating the elemental moments.

$$F \times l_{cp} = \int dF \times l$$

$$\rho g A y_c \times l_{cp} = \int \rho g y\, dA \times l = \rho g \sin\alpha \boxed{\int l^2 dA} = I_O$$

$$\rho g A y_c \times l_{cp} = \rho g \sin\alpha\, I_O$$

[Also, $I_O = I_c + A l_c^2$

$l_{cp} = \dfrac{y_{cp}}{\sin\alpha}$; $l_c = \dfrac{y_c}{\sin\alpha}$

$$\therefore\ \rho g A y_c \times \frac{y_{cp}}{\sin\alpha} = \rho g \sin\alpha \left[I_c + A \frac{y_c^2}{\sin^2\alpha} \right]$$

or

$$y_{cp} = \frac{I_c \sin^2\alpha}{A y_c} + y_c$$

Similar analysis for x_{cp} ⟹ See F.M. White.

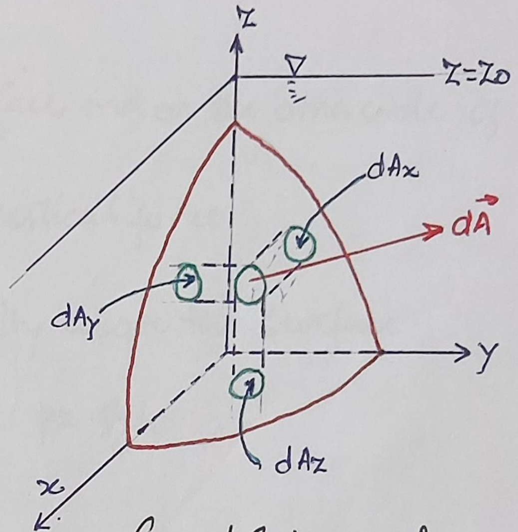
4. On Curved Submerged plane.

Complications:- Rr. force is normal to the surface area at each pt. Each elemental area would point in different direction. So integration would be $\underbrace{\int d\vec{A}}_{\text{Vector}}$ instead of dA.

Pressure fr. on $d\vec{A}$

$$d\vec{F} = -p\,d\vec{A}$$

'–' sign shows force acts opposite to area normal.



Curved Submerged Surface.

Resultant force.

$$\vec{F_R} = -\int_A p\,d\vec{A}$$

Also,

$$\vec{F_R} = \hat{i}\,F_{Rx} + \hat{j}\,F_{Ry} + \hat{k}\,F_{Rz}$$

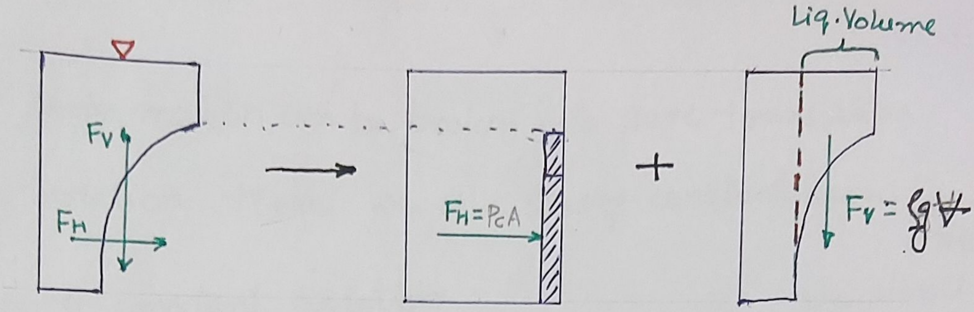
Comp. of $\vec{F_R}$ in +ve, x, y & z.

$$F_{Rx} \equiv \vec{F_R}\cdot\hat{i} = \int d\vec{F}\cdot\hat{i} = -\int_A p\,d\vec{A}\cdot\hat{i} = -\int_{A_x} p\,dA_x$$

where. dA_x = projection of $d\vec{A}$ on a plane ⊥ x axis.

∴ For F_{Rx} & F_{Ry} [magnitudes]

"The horizontal force and its location are the same as for an imaginary vertical plane surface of the same projected area..



Forces on curved Submerged Surface.

With atm. pr. at the free surface and on the other side of the curved surface. The net vertical force

$F_{R_z} = F_v$ = wt. of fluid directly above the surface.

$= \int p \, dA_z$ But $p = \rho g h$

$= \int \rho g h \, dA_z = \int \rho g \, d\forall$

$\rho g d\forall$ is the wt. of the differential cylinder of liq. above the element of surface area dA_z, extending a distance h from ~~above~~ the curved surface to the free surface.

$$\therefore \quad F_V = \int_{A_z} \rho g \, h \, dA_z = \int_{\forall} \rho g \, d\forall = \rho g \forall$$

$$F_H = P_c A \quad ; \quad F_V = \rho g \forall$$

where, P_c and A are the pr. at the centre and the area of the vertical plane. & $\forall$ is the volume of the fluid above the curved surface.

Pressure distribution in rigid body motion.

A rigid body motion can be broken into pure translation & pure rotation. Here we will study constant acen. and pure constant rotation.

[In exam don't use abbreviations without giving nomenclature first]

We already know that the forces due to pr. & gravity acting on a fluid particle of volume $d\forall$. are given by.

$$d\vec{F} = (-\nabla p + \rho\vec{g})\, d\forall$$

$$\text{or } \frac{d\vec{F}}{d\forall} = -\nabla p + \rho\vec{g} \quad -$$

From Newton's IInd law.

$$d\vec{F} = \vec{a}\, dm = \vec{a}\rho\, d\forall \quad \text{or } \frac{d\vec{F}}{d\forall} = \rho\vec{a}$$

$$-\nabla p + \rho\vec{g} = \rho\vec{a}$$

$$\text{or} \quad -\nabla p + \rho\vec{g}_{eff} = 0$$

[for $\vec{a}$ to be constant $\vec{g}_{eff} = \vec{g} - \vec{a}$.

V. Imp!:- Above eq. is a proof that lines of constt pr will be $\perp$ to the direction of $\vec{g}_{eff}$. For $\vec{a} = 0$, $\vec{g}_{eff} = \vec{g}$. Which gives us the eq. of free liquid surface. In other words a mathematical proof of the shape of a freely standing liq. surface.

In rectangular coordinates the compoment eq. are

$$-\frac{\partial p}{\partial x} + \rho g_x = \rho a_x \qquad x\text{-dir.}$$

$$-\frac{\partial p}{\partial y} + \rho g_y = \rho a_y \qquad y\text{-dir.}$$

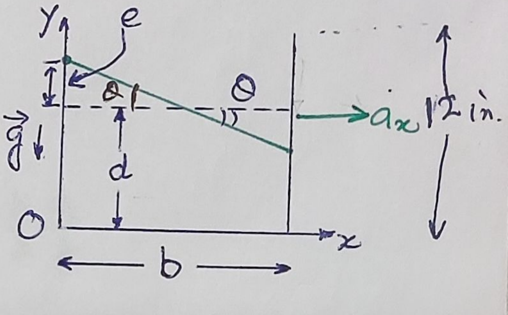
$$-\frac{\partial p}{\partial z} + \rho g_z = \rho a_z \qquad z\text{-dir.}$$

Problem:- You want to transport a fish tank in the back of your car. The tank is 12 in. X 24 in X 12 in. How much water can you leave in the tank and be reasonably sure that it will not spill over during the trip?

Assumptions:- Neglect sloshing due to jerks.

Find:- 1) Shape of free surface under constt. accm.

2) Allowable water depth d, to avoid spilling as a function of a_x & tank orientation.



3. Optimum tank orientation and recommended water depth.

Sol:

$$-\vec{\nabla} p + \rho \vec{g} = \rho \vec{a}$$

$$-\left(\hat{i}\frac{\partial p}{\partial x} + \hat{j}\frac{\partial p}{\partial y} + \hat{k}\frac{\partial p}{\partial z}\right) + \rho\left(\hat{i} g_x + \hat{j} g_y + \hat{k} g_z\right) = \rho\left(\hat{i} a_x + \hat{j} a_y + \hat{k} a_z\right)$$

Since $P \neq fn(z)$. $\partial P/\partial z = 0$, Also, $g_x = g_z = 0$, $g_y = -g$.

$a_y = a_z = 0$

$$\therefore -\hat{i}\frac{\partial p}{\partial x} - \hat{j}\frac{\partial p}{\partial y} - \hat{j}\rho g = \hat{i}\rho a_x$$

∴ The component eqs are.

$$\frac{\partial P}{\partial x} = -\rho a_x$$

$$\frac{\partial P}{\partial y} = -\rho g$$

Since $P = P(x, y)$

$$dp = \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy$$

$P =$ const. along the free surface, so $dp = 0$

$$0 = -\rho a_x dx - \rho g dy$$

$$\therefore \left.\frac{dy}{dx}\right)_{\text{free surf}} = -\frac{a_x}{g}$$

From the fig.

$$e = \frac{b}{2}\tan\theta = \frac{b}{2}\left(-\frac{dy}{dx}\right)_{\text{free surf}} = \frac{b}{2}\frac{a_x}{g}$$

Since we want e, to be smallest for a given a_x, the tank should be aligned so that b is small. Therefore, choose $b = 12$ in. instead of 24 in.

Now, $e = 6\frac{a_x}{g}$ in.

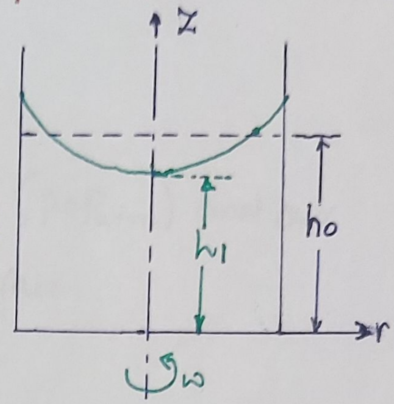
Also, $e = 12 - d$ in. [ht. of tank is 12 in.].

or $12 - d = 6\frac{a_x}{g}$ or $d_{max} = 12 - 6\frac{a_x}{g}$.

(20)

Rigid body motion with constt. angular speed.

Problem:- A cylindrical container partially filled with liq. is rotated at ω. The liquid rotates with the cyl as if the system were a rigid body. Determine the shape of the free surface.



Governing eq:

$$-\nabla p + \rho \vec{g} = \rho \vec{a}$$

Using cylindrical coordinate system. r, θ, z

$g_r = g_\theta = 0$; $g_z = -g$.

$$-\left(\hat{e}_r \frac{\partial p}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial p}{\partial \theta} + \hat{k}\frac{\partial p}{\partial z}\right) - \hat{k}\rho g = \rho\left(\hat{e}_r a_r + \hat{e}_\theta a_\theta + \hat{k} a_z\right)$$

Also, $a_\theta = a_z = 0$, $a_r = -\omega^2 r$

$$\therefore -\left(\hat{e}_r \frac{\partial p}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial p}{\partial \theta} + \hat{k}\frac{\partial p}{\partial z}\right) = -\hat{e}_r \omega^2 r \rho + \hat{k}\rho g$$

The component eq. are:-

$$\frac{\partial p}{\partial r} = \rho\omega^2 r \qquad \frac{\partial p}{\partial \theta} = 0 \qquad \frac{\partial p}{\partial z} = -\rho g$$

Since $p \neq p(\theta)$ but $p = p(r, z)$.

$$dp = \left.\frac{\partial p}{\partial r}\right|_z dr + \left.\frac{\partial p}{\partial z}\right|_r dz.$$

Then $\quad dp = \rho\omega^2 r\, dr - \rho g\, dz.$

To obtain the pr. diff. between a reference pt. (r_1, z_1) where pr. is p_1, and the arbitrary pt. (r, z) where pr. is p, we integrate.

$$\int_{p_1}^{p} dp = \int_{r_1}^{r} \rho\omega^2 r\, dr - \int_{z_1}^{z} \rho g\, dz.$$

$$p - p_1 = \frac{\rho\omega^2}{2}(r^2 - r_1^2) - \rho g(z - z_1)$$

Taking the reference pt. on cyl. axis at free surface.

$$P_1 = P_{atm},\ r_1 = 0,\ z_1 = h_1$$

Then

$$P - P_{atm} = \frac{\rho\omega^2 r^2}{2} - \rho g(z - h_1)$$

Since the pr. is constt. at free surface ($P = P_{atm}$) and our objective is to find eq. of the free surface.

$$\therefore \quad 0 = \frac{\rho\omega^2 r^2}{2} - \rho g(z - h_1)$$

$$\text{or } Z = h_1 + \frac{(\omega r)^2}{2g}$$

We can also find h_1 in terms of h_0. Using the fact that the liq. volume must remain constant.

$$\forall_{\text{no rotation}} = \forall_{\text{with rotation}}$$

$$\pi R^2 h_0 = \int_0^R \int_0^z 2\pi r\,dz\,dr = \int_0^R 2\pi Z r\,dr$$

$$= \int_0^R 2\pi\left(h_1 + \frac{\omega^2 r^2}{2g}\right) r\,dr$$

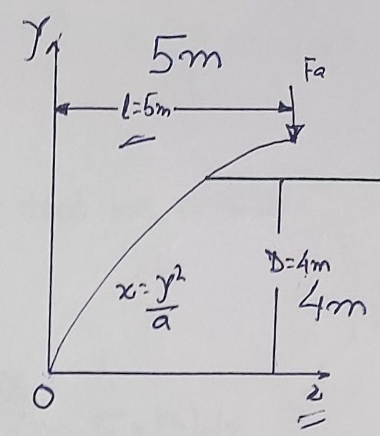
$$= 2\pi\left[\frac{h_1 r^2}{2} + \frac{\omega^2 r^4}{8g}\right]_0^R$$

$$\text{or} \quad \pi R^2 h_0 = \pi\left[h_1 R^2 + \frac{\omega^2 R^4}{4g}\right] \quad \text{or} \quad h_1 = h_0 - \frac{(\omega R)^2}{4g}$$

Also, $Z = h_0 - \dfrac{(\omega R)^2}{4g} + \dfrac{(\omega r)^2}{2g}$.

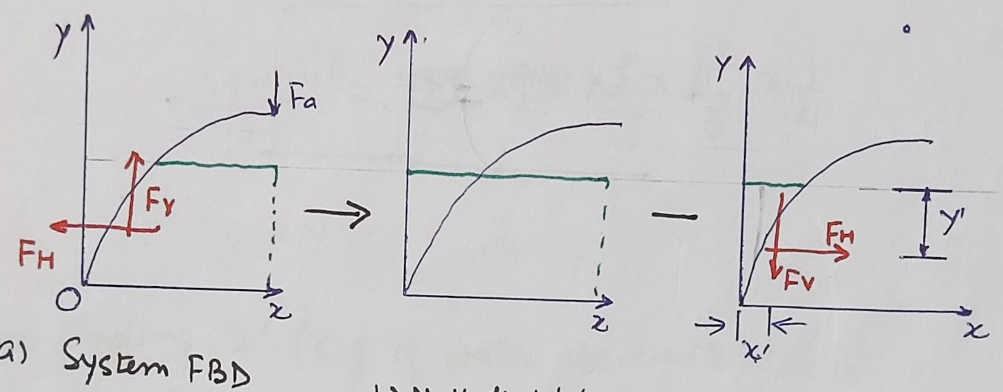
ω_{max} such that $h_1 > 0$; $= 2\sqrt{g h_0}/R$

1. Gate is hinged at O. and has constant width, w = 5m. The eq. of surface is $x = Y^2/a$ where a = 4m. The depth of water is D = 4m. Find Fa, required to maintain the gate in equilibrium if the wt. of the gate is neglected.



Given:- w = 5m, D = 4m.

eq. of surface in xy plane, $x = y^2/a$, a = 4m.



(a) System FBD

b) Null fluid force

c) Fluid force.

An equivalent system of fig (a). is suggested in fig. (b) & (c). The mag. & location of Fv are given by the weight and location of the centroid of the fluid above the gate; similarly for FH are given by that of the force on an equivalent vertical flat plate.

Governing Eq:: $F_H = P_c A$; $Y' = Y_c + \frac{I_c}{A Y_c}$; $F_V = \rho g \forall$, x': water c.g.

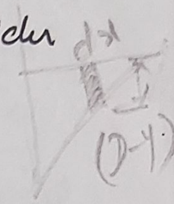
$Y_c = h_c = D/2$, $A = DW$, $I_c = WD^3/12$.

$$F_H = P_c A = \rho g h_c A = \rho g \frac{D}{2} D W =$$

$$= 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times \frac{(4m)^2}{2} \times 5m = 392 kN$$

$$y' = y_c + \frac{I_c}{A y_c} = \frac{D}{2} + \frac{wD^3/12}{wD\,D/2} = \frac{D}{2} + \frac{D}{6} = \frac{2D}{3} = 2.67\,m.$$

For F_v, we find wt. of water above. For this we consider a differential volume $(D-y)\,w\,dx$.



$$F_v = \rho g \forall = \rho g \int_0^{D^2/a} (D-y)\,w\,dx = \rho g w \int_0^{D^2/a} (D - \sqrt{a}\,x^{1/2})\,dx$$

$$= \rho g w \left[Dx - \frac{2}{3}\sqrt{a}\,x^{3/2} \right]_0^{D^2/a} = \rho g w \left[\frac{D^3}{a} - \frac{2}{3}\sqrt{a}\,\frac{D^3}{a^{3/2}} \right]$$

$$= \frac{\rho g w D^3}{3a} = 999 \times 9.81 \times 5 \times \frac{4^3}{3} \times \frac{1}{4}$$

$= 261\,kN.$

For finding x' (c.g of water above gate)

$$x' F_v = \rho g \int_0^{D^2/a} x(D-y)\,w\,dx = \rho g w \int_0^{D^2/a} (xD - \sqrt{a}\,x^{3/2})\,dx$$

$$= \frac{\rho g w D^5}{10 a^2}$$

$$\text{or } x' = \frac{\rho g w D^5}{10 a^2 F_v} = \frac{3D^2}{10a} = 1.2\,m$$

For finding F_a we take moments about O.

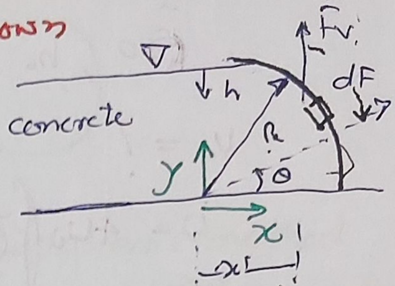
$$\sum M_O = -l F_a + x' F_v + (D - y') F_H = 0$$

$$\text{or } F_a = \frac{1}{l}\left[x' F_v + (D-y') F_H \right]$$

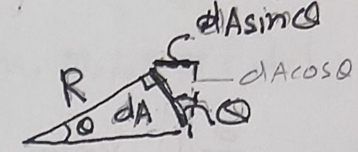
$$= \frac{1}{5}\left[1.2 \times 261 \times 10^3 + (4 - 2.67) \times 392 \times 10^3 \right]$$

$$F_a = 167\,kN$$

Prob: Liq. concrete is poured into the form shown ($R = 0.313$ m). Width is 4.25 m normal to dig. Find F_v and its line of action.



$$P = \rho g h$$



$$F_v = \int P(dA_y) = \int \rho g h \, dA \sin\theta \; ; \; dA = w R \, d\theta$$

$$h = R - y = R - R\sin\theta$$

$$\therefore F_v = \int_0^{\pi/2} \rho g \, R(1-\sin\theta)\sin\theta \, w R \, d\theta = \rho g R^2 w \int_0^{\pi/2} (\sin\theta - \sin^2\theta)\, d\theta$$

$$= \rho g R^2 w (1 - \pi/4) \qquad \left[\rho g\right]_{concrete} = \overbrace{2.5}^{S.G_{concrete}} \times 1000 \times 9.8$$

$$= 2.19 \text{ kN}$$

$$x' F_v = \left[\rho g R^2 w \int_0^{\pi/2} (\sin\theta - \sin^2\theta)\, x \, d\theta\right] \qquad x = R\cos\theta$$

$$= \rho g R^3 w \int_0^{\pi/2} (\sin\theta\cos\theta - \sin^2\theta\cos\theta)\, d\theta = \rho g R^3 w \left[\frac{\sin^2\theta}{2} - \frac{\sin^3\theta}{3}\right]_0^{\pi/2}$$

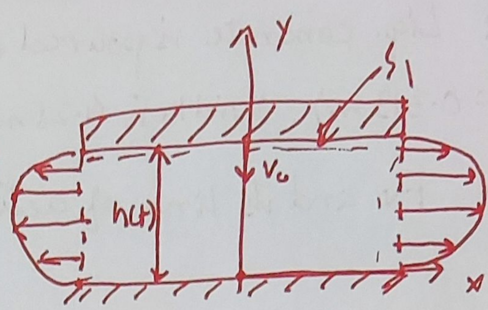
$$x' F_v = \rho g R^3 w \left[\frac{1}{2} - \frac{1}{3}\right] = \frac{\rho g R^3 w}{6}$$

$$x' = 0.243 \text{ m}$$

$t=0$, h_0 , B

$U_0 = ?$

$$U = 4U_0\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right]$$



$$\frac{\partial}{\partial t}\oint \rho\, dV + \oint \rho\, \vec{V_1}\cdot d\vec{S} = 0$$

0

$\rho = C$

$$\rho\,\frac{\partial}{\partial t}\int_0^h B\, dy$$

$$\rho\, B\frac{dh}{dt} + \underline{\qquad\qquad} -$$

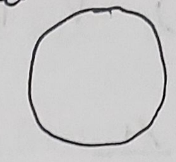
$-V_0$

$$\vec{V_0}\cdot\hat{n}\Big|_{S_1} = \vec{V_w}\cdot\hat{n} = -V_0$$

$\iint$

Dimensional Analysis.

Step 1. List all dimensional parameters involved



$$F = f(\rho, V, D, \mu)$$

$$g(F, \rho, V, D, \mu) = 0$$

Buckingham's method (1914)

Step 2. Select a set of fundamental/primary dim. M, L, t or F, L, t

Step 3. List the dim of all parameters.

Step 4. Select a set of r dimensional parameters that include all the primary dim.

Step 5. Set up dimensional eq. combining the parameters selected in Step 4. with each of the other parameter in turn. to form dimensionless group.

Step. Check to see if each group is dimensionless.

(1) $F \quad V \quad D \quad \rho \quad \mu$

(2) $M \quad L \quad t$

(3)

F	V	D	ρ	μ
$\frac{ML}{t^2}$	$\frac{L}{t}$	L	$\frac{M}{L^3}$	$\frac{M}{Lt}$

(4) $\rho \quad V \quad D$

(5) $\Pi_1 = \rho^a V^b D^c F$ or $\left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \left(\frac{ML}{t^2}\right) = M^0 L^0 T^0$

$$\Pi_1 = \frac{F}{\rho V^2 D^2}$$

$$\Pi_2 = \rho^d V^e D^f \mu \Rightarrow \left(\frac{M}{L^3}\right)^d \left(\frac{L}{t}\right)^e (L)^f \left(\frac{M}{Lt}\right) = M^0 L^0 t^0$$

$$\Pi_2 = \frac{\mu}{\rho V D} \qquad \Pi_1 = \phi \, \Pi_2$$

Guide lines for choosing repeating variables (m)

1. To start with, choose (m) equal to the no. of primary dimensions (P) [i.e (MLt or FLt)] of the problem

2. m should be ~~always~~ less than or equal to the no. of the primary dime. ∴ $m \leq P$.

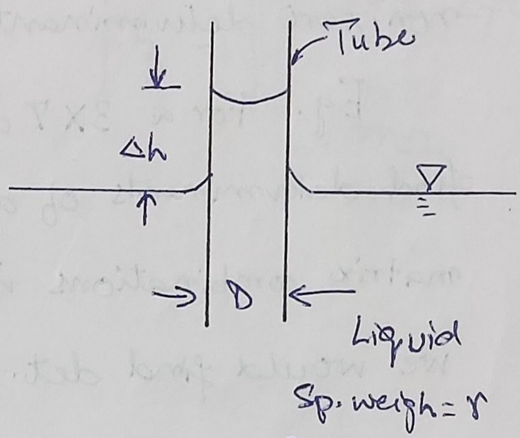
3. The value of (m) can be established with certainity by determining the rank of the dimensional matrix; that rank is m.

4. The Π groups are independent but not unique. If different set of repeating ~~variables~~/parameters are used, different groups result.

5. Based on experience, viscosity should appear in only one dimensionless parameter. Therefore μ should not be chosen as a repeating parameter.

6). If given the choice, it usually works best to choose $\rho\,(M/L^3)$, speed $V\,(L/t)$ and characteristic length $L\,(L)$ as repeating parameters. Experience shows this generally leads to Π groups that are suitable for correlating a wide range of exp. data. Also, ρ, V, L are fairly easy to measure.

Problem. Capillary Effect:
(Using dimensional matrix)



$$\Delta h = f(D, \gamma, \sigma)$$

1. $\Delta h \quad D \quad \gamma \quad \sigma \qquad n = 4$

2, Use both M, L, t & F, L, t

(3) a) M, L, t

Δh	D	γ	σ
L	L	$\frac{M}{L^2t^2}$	$\frac{M}{t^2}$

$r = 3$

b) F, L, t

Δh	D	γ	σ
L	L	$\frac{F}{L^3}$	$\frac{F}{L}$

$r = 2$.

Rank of a matrix:

The rank 'r' of any matrix is defined to be the size of the largest square submatrix that has a non-zero determinant.

Eg. For a 3×7 dimensional matrix, we would first find determinants of diff. 3×3 matrices. If det. for all 3×3 matrix combinations is zero, we conclude $r < 3$. Next, we we would find det. of 2×2 matrix combinations.

Reading Assignment: Complete the following sections yourself

Similarity

Geometric

Kinematic

Dynamic

Buoyancy and stability.

	Δh	D	r	σ
M	0	0	1	1
L	1	1	-2	0
t	0	0	-2	-2

	Δh	D	r	σ
F	0	0	1	1
L	1	1	-3	-1

The rank of a matrix is equal to the order of its largest non zero determinant.

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 0 & -2 & -2 \end{vmatrix} = 0-(1)(-2)+(1)(-2) = 0$$

$$\begin{vmatrix} -2 & 0 \\ -2 & -2 \end{vmatrix} = 4 \neq 0 \quad \therefore m = 2$$

$$m \neq r$$

$$\begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix} = -1+3 = 2 \neq 0$$

$$\therefore m = 2$$

$$m = r$$

(4) Choose D, r as repeating ~~variables~~ parameters.

$$\Pi_1 = D^a r^b \Delta h$$

$$(L)^a \left(\frac{M}{L^2 t^2}\right)^b (L) = M^0 L^0 t^0$$

$$\Pi_1 = \Delta h / D$$

$$\Pi_2 = D^c r^d \sigma$$

$$= (L)^c \left(\frac{M}{L^2 t^2}\right)^d \frac{M}{t^2} = M^0 L^0 t^0$$

$$= \sigma / D^2 r$$

$$\Pi_1 = D^e r^f \Delta h$$

$$= (L)^e \left(\frac{F}{L^3}\right)^f L = F^0 L^0 t^0$$

$$\Pi_1 = \Delta h / D$$

$$\Pi_2 = D^g r^h \sigma$$

$$= (L)^g \left(\frac{F}{L^3}\right)^h \frac{F}{L} = F^0 L^0 t^0$$

$$\Pi_2 = \sigma / D^2 r$$

$$\Pi_1 = f(\Pi_2)$$

$$\text{or } \Delta h / D = f\left(\frac{\sigma}{D^2 r}\right)$$

Drag force F depends on various parameters like the diameter D, fluid viscosity μ, fluid velocity V & fluid density ρ. Find out the dimensionless group.

Solution
$$F = f(D, \mu, V, \rho)$$

Choose appropriate parameters and use primary dimension as M L T

∴ Select repeating parameters as V, D & ρ

For π_1

∴ $V\,D\,\rho\,F$

$$\left(\frac{L}{T}\right)^a (L)^b \left(\frac{M}{L^3}\right)^c \left(\frac{ML}{T^2}\right) = M^0 L^0 T^0$$

∴ $L \rightarrow \quad a + b - 3c + 1 = 0$

$M \rightarrow \quad c + 1 = 0 \Rightarrow c = -1$

$T \rightarrow \quad -a - 2 = 0 \Rightarrow a = -2$

∴ $a + b - 3c + 1 = 0$

$-2 + b - 3(-1) + 1 = 0$

$b = -2$

$$\pi_1 = V^{-2} D^{-2} \rho^{-1} \cdot F$$

$$\pi_1 = \frac{F}{\rho V^2 D^2}$$

For π_2 $\quad V\,D\,\rho\,\mu$

$$\left(\frac{L}{T}\right)^a (L)^b \left(\frac{M}{L^3}\right)^c \left(\frac{M}{LT}\right) = M^0 L^0 T^0$$

$L \rightarrow a + b - 3c - 1 = 0$

$M \rightarrow c + 1 = 0 \Rightarrow c = -1$

$T \rightarrow$ ~~$-a + 1 = 0 \Rightarrow a = 1$~~ $-a - 1 = 0 \Rightarrow a = -1$